

Comparison theorems for summability methods of sequences of fuzzy numbers

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Abstract: In this study we compare Cesàro and Euler weighted mean methods of summability of sequences of fuzzy numbers with Abel and Borel power series methods of summability of sequences of fuzzy numbers. Also some results dealing with series of fuzzy numbers are obtained.

1 Introduction

It is well known that the facts that human being met in the natural world are generally complex and inexact. Complexity and inexactness of real-world events often stems from uncertain nature of the parameters and from vague status of the underlying objects. Realizing that uncertainty is ubiquitous and essential in complex systems, researchers designed many uncertainty theories such as probability theory, evidence theory, fuzzy set theory to cope with problems of vagueness. Considered as the recent one, fuzzy set theory was introduced by Zadeh [23] in 1965 and advanced in many branches of science and engineering. In mathematics, in connection with the concepts of sequences Matloka [8] introduced bounded and convergent sequences of fuzzy numbers and Nanda [10] showed that every Cauchy sequence of fuzzy numbers is convergent. Different classes of sequences of fuzzy numbers were introduced and convergence properties of sequences and series of fuzzy numbers were given [1, 9, 12, 13, 19]. Besides, with the purpose of handling divergent sequences, various summability methods of sequences of fuzzy numbers were introduced and Tauberian conditions which guarantee the convergence of summable sequences were given [2, 4, 11, 14, 18]. In particular, Cesàro, Euler weighted mean methods of summability and Abel, Borel power series methods of summability for sequences of fuzzy numbers have also been studied recently and corresponding Tauberian theorems have been proved [5, 6, 16, 17, 20–22].

The main goal of this paper is to compare Cesàro and Euler summability methods of sequences of fuzzy numbers with Abel and Borel summability methods, respectively. To achieve this goal, in Section 3 we give an optimal bound for Cesàro summable sequences of fuzzy numbers and prove a comparison theorem between Cesàro and Abel methods of summability of sequences of fuzzy numbers. A Mertens' type result concerning multiplication of series of fuzzy numbers is also obtained. In section 4 firstly we show that Euler summability method E_p becomes stronger in summing up divergent sequences of fuzzy numbers as the order p increases and then prove that E_p convergence of a sequence of fuzzy numbers implies Borel convergence. Finally in Section 5, as results of comparisons made in Section 3-4, some Tauberian theorems for Abel and Borel methods of summability of sequences of fuzzy numbers have been extended to Cesàro and Euler summability methods.

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2 Preliminaries

A *fuzzy number* is a fuzzy set on the real axis, i.e. u is normal, fuzzy convex, upper semi-continuous and $\text{supp } u = \overline{\{t \in \mathbb{R} : u(t) > 0\}}$ is compact [23]. We denote the space of fuzzy numbers by E^1 . α -level set $[u]_\alpha$ of $u \in E^1$ is defined by

$$[u]_\alpha := \begin{cases} \{t \in \mathbb{R} : u(t) \geq \alpha\} & , \quad \text{if } 0 < \alpha \leq 1, \\ \overline{\{t \in \mathbb{R} : u(t) > \alpha\}} & , \quad \text{if } \alpha = 0. \end{cases}$$

Let $u, v \in E^1$ and $k \in \mathbb{R}$. The addition and scalar multiplication are defined by

$$[u + v]_\alpha = [u]_\alpha + [v]_\alpha = [u_\alpha^-, v_\alpha^-, u_\alpha^+ + v_\alpha^+], [ku]_\alpha = k[u]_\alpha$$

where $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, for all $\alpha \in [0, 1]$.

Lemma 2.1. [3] *The following statements hold:*

- (i) $\bar{0} \in E^1$ is neutral element with respect to $+$, i.e., $u + \bar{0} = \bar{0} + u = u$ for all $u \in E^1$.
- (ii) With respect to $\bar{0}$, none of $u \neq \bar{r}$, $r \in \mathbb{R}$ has opposite in E^1 .
- (iii) For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$ and any $u \in E^1$, we have $(a + b)u = au + bu$. For general $a, b \in \mathbb{R}$, the above property does not hold.
- (iv) For any $a \in \mathbb{R}$ and any $u, v \in E^1$, we have $a(u + v) = au + av$.
- (v) For any $a, b \in \mathbb{R}$ and any $u \in E^1$, we have $a(bu) = (ab)u$.

The metric D on E^1 is defined as

$$D(u, v) := \sup_{\alpha \in [0, 1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}.$$

Proposition 2.2. [3] *Let $u, v, w, z \in E^1$ and $k \in \mathbb{R}$. Then,*

- (i) (E^1, D) is a complete metric space.
- (ii) $D(ku, kv) = |k|D(u, v)$.
- (iii) $D(u + v, w + v) = D(u, w)$.
- (iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$.
- (v) $|D(u, \bar{0}) - D(v, \bar{0})| \leq D(u, v) \leq D(u, \bar{0}) + D(v, \bar{0})$.

A sequence (u_n) of fuzzy numbers is said to be convergent to $\mu \in E^1$ if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_n, \mu) < \varepsilon$ for all $n \geq n_0$. We mean that sequence (u_n) converges to μ by $u_n \rightarrow \mu$.

Definition 2.3. [7] *Let (u_k) be a sequence of fuzzy numbers. Then the expression $\sum u_k$ is called a series of fuzzy numbers. Denote $s_n = \sum_{k=0}^n u_k$ for all $n \in \mathbb{N}$. If the sequence (s_n) converges to a fuzzy number u , then we say that the series $\sum u_k$ of fuzzy numbers converges to u and write $\sum u_k = u$ which implies as $n \rightarrow \infty$ that*

$$\sum_{k=0}^n u_k^-(\lambda) \rightarrow u^-(\lambda) \text{ and } \sum_{k=0}^n u_k^+(\lambda) \rightarrow u^+(\lambda)$$

uniformly in $\lambda \in [0, 1]$. Conversely, if the series $\sum_k u_k^-(\lambda) = u^-(\lambda)$ and $\sum_k u_k^+(\lambda) = u^+(\lambda)$ converge uniformly in λ , then $u = \{(u^-(\lambda), u^+(\lambda)) : \lambda \in [0, 1]\}$ defines a fuzzy number such that $u = \sum u_k$. We say otherwise the series of fuzzy numbers diverges.

Remark 2.4. Let (u_n) be a sequence of fuzzy numbers. If (x_n) is a sequence of non-negative real numbers, then

$$\sum_{k=0}^n x_k \sum_{m=0}^k u_m = \sum_{m=0}^n u_m \sum_{k=m}^n x_k$$

holds by (iii) and (iv) of Lemma 2.1.

Theorem 2.5. [15] If $\sum u_n$ and $\sum v_n$ converge, then $D(\sum u_n, \sum v_n) \leq \sum D(u_n, v_n)$.

Theorem 2.6. [15] If $\sum D(u_k, \bar{0}) < \infty$, then the series $\sum u_k$ is convergent.

Cesàro, Euler weighted mean methods of summability and Abel, Borel power series methods of summability for sequences of fuzzy numbers have been defined recently as the following:

Definition 2.7. [14] Let (u_n) be a sequence of fuzzy numbers and let sequence of arithmetic means of (u_n) be defined by $\sigma_n = \frac{1}{n+1} \sum_{k=0}^n u_k$. We say that sequence (u_n) is Cesàro summable to fuzzy number μ if $\lim_{n \rightarrow \infty} \sigma_n = \mu$.

Definition 2.8. [22] Let (u_n) be a sequence of fuzzy numbers. The Euler means of (u_n) is defined by

$$t_n^p = \frac{1}{(p+1)^n} \sum_{k=0}^n \binom{n}{k} p^{n-k} u_k \quad (p > 0).$$

We say that (u_n) is E_p summable to a fuzzy number μ if $\lim_{n \rightarrow \infty} t_n^p = \mu$.

Definition 2.9. [20] A sequence (u_n) of fuzzy numbers is said to be Abel summable to a fuzzy number μ if the series $\sum_{n=0}^{\infty} u_n x^n$ converges for all $x \in (0, 1)$ and

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = \mu.$$

Definition 2.10. [21] A sequence (u_n) of fuzzy numbers is said to be Borel summable to μ if the series $\sum_{n=0}^{\infty} \frac{x^n}{n!} u_n$ converges for all $x \in (0, \infty)$ and

$$\lim_{x \rightarrow \infty} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} u_n = \mu.$$

3 Comparison between Cesàro and Abel methods of summability of sequences of fuzzy numbers

In the following theorem we give an optimal bound for Cesàro summable sequences of fuzzy numbers.

Theorem 3.1. If sequence (u_n) of fuzzy numbers is Cesàro summable, then best possible bound for (u_n) is $D(u_n, \bar{0}) = o(n)$.

Proof. Let sequence (u_n) of fuzzy numbers be Cesàro summable to a fuzzy number μ . Then sequence of Cesàro means $\sigma_n = \frac{1}{n+1} \sum_{k=0}^n u_k$ converges to μ . From Proposition 2.2 we have

$$D(u_n, \bar{0}) = D\left(\sum_{k=0}^n u_k, \sum_{k=0}^{n-1} u_k\right) = D((n+1)\sigma_n, n\sigma_{n-1}) \leq nD(\sigma_n, \sigma_{n-1}) + D(\sigma_n, \bar{0})$$

and, by dividing both sides with n , we get

$$\frac{D(u_n, \bar{0})}{n} \leq D(\sigma_n, \sigma_{n-1}) + \frac{D(\sigma_n, \bar{0})}{n}.$$

Since (σ_n) is a convergent sequence, by limiting both sides we conclude $D(u_n, \bar{0}) = o(n)$.

Now we shall show that estimate $D(u_n, \bar{0}) = o(n)$ is best possible. We prove by contradiction. Let estimate $D(u_n, \bar{0}) = o\left(\frac{n}{\lambda_n}\right)$ be best possible bound for Cesàro summable sequences (u_n) of fuzzy numbers, where (λ_n) is a sequence of real numbers with $0 < \lambda_n \neq O(1)$. Then there exists a subsequence (λ_{n_k}) of (λ_n) such that $n_{k+1} \geq n_k + 2$ and $\lambda_{n_k} \uparrow \infty$. Then consider the sequence of fuzzy numbers (u_n) defined by:

$$u_{n_k}(t) = \begin{cases} t - \frac{n_k}{\sqrt{\lambda_{n_k}}}, & \frac{n_k}{\sqrt{\lambda_{n_k}}} \leq t \leq \frac{n_k}{\sqrt{\lambda_{n_k}}} + 1 \\ 2 - t + \frac{n_k}{\sqrt{\lambda_{n_k}}}, & \frac{n_k}{\sqrt{\lambda_{n_k}}} + 1 \leq t \leq \frac{n_k}{\sqrt{\lambda_{n_k}}} + 2 \\ 0, & \text{otherwise,} \end{cases}$$

$$u_{n_k+1}(t) = \begin{cases} t + \frac{n_k}{\sqrt{\lambda_{n_k}}}, & -\frac{n_k}{\sqrt{\lambda_{n_k}}} \leq t \leq 1 - \frac{n_k}{\sqrt{\lambda_{n_k}}} \\ 2 - t - \frac{n_k}{\sqrt{\lambda_{n_k}}}, & 1 - \frac{n_k}{\sqrt{\lambda_{n_k}}} \leq t \leq 2 - \frac{n_k}{\sqrt{\lambda_{n_k}}} \\ 0, & \text{otherwise,} \end{cases}$$

for $n = n_k, n = n_k + 1$ and

$$u_n(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

for $n \neq \{n_k, n_k + 1\}$. Then α -level set of (u_n) is

$$[u_{n_k}]_\alpha = \left[\alpha + \frac{n_k}{\sqrt{\lambda_{n_k}}}, 2 - \alpha + \frac{n_k}{\sqrt{\lambda_{n_k}}} \right], \quad [u_{n_k+1}]_\alpha = \left[\alpha - \frac{n_k}{\sqrt{\lambda_{n_k}}}, 2 - \alpha - \frac{n_k}{\sqrt{\lambda_{n_k}}} \right]$$

for $n = n_k, n = n_k + 1$ and $[u_n]_\alpha = [\alpha, 2 - \alpha]$ for $n \neq \{n_k, n_k + 1\}$. So α -level set of Cesàro means (σ_n) is

$$[\sigma_{n_k}]_\alpha = \left[\alpha + \frac{n_k}{(n_k + 1)\sqrt{\lambda_{n_k}}}, 2 - \alpha + \frac{n_k}{(n_k + 1)\sqrt{\lambda_{n_k}}} \right]$$

for $n = n_k$ and $[\sigma_n]_\alpha = [\alpha, 2 - \alpha]$ for $n \neq n_k$. Thus we conclude that sequence (u_n) is Cesàro summable to fuzzy number

$$\mu(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 \leq t \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

However we have

$$\frac{\lambda_{n_k}}{n_k} D(u_{n_k}, \bar{0}) = \frac{\lambda_{n_k}}{n_k} \sup_{\alpha \in [0,1]} \max \left\{ \left| \alpha + \frac{n_k}{\sqrt{\lambda_{n_k}}} \right|, \left| 2 - \alpha + \frac{n_k}{\sqrt{\lambda_{n_k}}} \right| \right\} = \frac{2\lambda_{n_k}}{n_k} + \sqrt{\lambda_{n_k}},$$

which contradicts with the assumption $D(u_n, \bar{0}) = o\left(\frac{n}{\lambda_n}\right)$. The proof is completed. \square

Now we prove a theorem dealing with multiplication of infinite series of fuzzy numbers, which is analogous to Mertens theorem that in classical analysis.

Theorem 3.2. Let $\sum_{n=0}^{\infty} u_n$ be a convergent series of fuzzy numbers. If $\sum_{n=0}^{\infty} x_n$ is a convergent series with non-negative real terms, then

$$\left(\sum_{n=0}^{\infty} x_n \right) \left(\sum_{n=0}^{\infty} u_n \right) = \sum_{n=0}^{\infty} \sum_{k=0}^n u_k x_{n-k}.$$

Proof. Let $\sum_{n=0}^{\infty} u_n$ be a convergent series of fuzzy numbers and $\sum_{n=0}^{\infty} x_n$ be a convergent series with non-negative real terms. Then there exist $U \in E^1$ and $X \in \mathbb{R}$ such that $U_n = \sum_{k=0}^n u_k \rightarrow U$ and $X_n = \sum_{k=0}^n x_k \rightarrow X$ are satisfied. Hence for given any $\varepsilon > 0$

- (i) there exists $n_0 \in \mathbb{N}$ such that $D(U_n, U) < \frac{\varepsilon}{3(X+1)}$ whenever $n > n_0$,
- (ii) there exists $n_1 \in \mathbb{N}$ such that $x_n < \frac{\varepsilon}{3\left\{(n_0+1) \max_{0 \leq k \leq n_0} \{D(U_k, U)\} + 1\right\}}$ whenever $n > n_1$,
- (iii) there exists $n_2 \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} x_k < \frac{\varepsilon}{3(D(U, 0)+1)}$ whenever $n > n_2$.

On the other hand by Remark 2.4 we have

$$\sum_{n=0}^m \sum_{k=0}^n u_k x_{n-k} = \sum_{n=0}^m \sum_{k=0}^n x_k u_{n-k} = \sum_{k=0}^m x_k \sum_{n=k}^m u_{n-k} = \sum_{k=0}^m x_k \sum_{n=0}^{m-k} u_n = \sum_{k=0}^m x_k U_{m-k}.$$

Since

$$\begin{aligned} D\left(\sum_{n=0}^m \sum_{k=0}^n u_k x_{n-k}, XU\right) &= D\left(\sum_{k=0}^m x_k U_{m-k}, \sum_{k=0}^{\infty} x_k U\right) \\ &= D\left(\sum_{k=0}^m x_k U_{m-k}, \sum_{k=0}^m x_k U + \sum_{k=m+1}^{\infty} x_k U\right) \\ &\leq D\left(\sum_{k=0}^m x_k U_{m-k}, \sum_{k=0}^m x_k U\right) + D\left(\sum_{k=m+1}^{\infty} x_k U, \bar{0}\right) \\ &\leq \sum_{k=0}^{m-n_0-1} x_k D(U_{m-k}, U) + \sum_{k=m-n_0}^m x_k D(U_{m-k}, U) + D\left(\sum_{k=m+1}^{\infty} x_k U, \bar{0}\right), \end{aligned}$$

we get

$$D\left(\sum_{n=0}^m \sum_{k=0}^n u_k x_{n-k}, XU\right) < \varepsilon$$

whenever $m > \max\{n_0 + n_1, n_2\}$, and this completes the proof. \square

Theorem 3.3. *If sequence (u_n) of fuzzy numbers is Cesàro summable to fuzzy number μ , then (u_n) is Abel summable to μ .*

Proof. Let (u_n) be Cesàro summable to a fuzzy number μ . We want to show that series $\sum u_n x^n$ of fuzzy numbers is convergent for $x \in (0, 1)$, and

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = \mu.$$

From Theorem 3.1 we have $D(u_n, \bar{0}) = o(n)$ and as result we get

$$\sum_{n=0}^{\infty} D(u_n x^n, \bar{0}) \leq \sum_{n=0}^{\infty} D(u_n, \bar{0}) x^n \leq \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

where $x \in (0, 1)$. So by Theorem 2.6, series $\sum u_n x^n$ of fuzzy numbers is convergent for $x \in (0, 1)$. Besides, from Theorem 3.2 we get

$$(1-x) \sum_{n=0}^{\infty} u_n x^n = (1-x)^2 \frac{1}{1-x} \sum_{n=0}^{\infty} u_n x^n = (1-x)^2 \left(\sum_{n=0}^{\infty} x^n \right) \left(\sum_{n=0}^{\infty} u_n x^n \right)$$

$$= (1-x)^2 \sum_{n=0}^{\infty} s_n x^n = (1-x)^2 \sum_{n=0}^{\infty} (n+1) \sigma_n x^n.$$

At this point we recall the power series method (J, p) introduced by Sefa and Çanak [11]. Since sequence (σ_n) of Cesàro means converges to μ and summability method $(J, n+1)$ is regular we have

$$\lim_{x \rightarrow 1^-} (1-x)^2 \sum_{n=0}^{\infty} (n+1) \sigma_n x^n = \mu,$$

from which we conclude

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = \mu.$$

□

However an Abel summable sequence of fuzzy number does not have to be Cesàro summable, which can be seen by following example.

Example 3.4. Consider sequence $u = (u_n)$ of fuzzy numbers such that

$$u_n(t) = \begin{cases} 2 \sqrt[n]{t + (-1)^{n+1}n}, & \text{if } (-1)^n n \leq t \leq (-1)^n n + \frac{1}{2^n}, \\ 1, & \text{if } (-1)^n n + \frac{1}{2^n} \leq t \leq (-1)^n n + 2 - \frac{1}{2^n}, \\ 2 \sqrt[n]{(-1)^n n + 2 - t}, & \text{if } (-1)^n n + 2 - \frac{1}{2^n} \leq t \leq (-1)^n n + 2, \\ 0, & \text{otherwise} \end{cases}$$

for $n \geq 1$ and $u_0 = \bar{1}$. Since

$$\begin{aligned} \sum_{n=0}^{\infty} u_n^-(\alpha) x^n &= \sum_{n=0}^{\infty} \left\{ (-1)^n n + \left(\frac{\alpha}{2}\right)^n \right\} x^n = \frac{-x}{(1+x)^2} + \frac{2}{2-\alpha x} & (0 < x < 1) \\ \sum_{n=0}^{\infty} u_n^+(\alpha) x^n &= \sum_{n=0}^{\infty} \left\{ (-1)^n n + 2 - \left(\frac{\alpha}{2}\right)^n \right\} x^n = \frac{-x}{(1+x)^2} + \frac{2}{1-x} - \frac{2}{2-\alpha x} & (0 < x < 1) \end{aligned}$$

converges uniformly in α , series $\sum u_n x^n$ is convergent by Definition 2.3. Then considering the fuzzy number μ , where $[\mu]_{\alpha} = [0, 2]$, we get

$$\begin{aligned} D \left((1-x) \sum u_n x^n, \mu \right) &= \sup_{\alpha \in [0,1]} \max \left\{ \left| \frac{-(1-x)x}{(1+x)^2} + \frac{2(1-x)}{2-\alpha x} \right|, \left| \frac{-(1-x)x}{(1+x)^2} + \frac{2(1-x)}{1-x} - \frac{2(1-x)}{2-\alpha x} - 2 \right| \right\} \\ &= \sup_{\alpha \in [0,1]} \left| \frac{-(1-x)x}{(1+x)^2} - \frac{2(1-x)}{2-\alpha x} \right| = \frac{(1-x)x}{(1+x)^2} + \frac{2(1-x)}{2-\alpha x} \end{aligned}$$

and so $\lim_{x \rightarrow 1^-} (1-x) \sum u_n x^n = \mu$. Hence sequence (u_n) of fuzzy numbers is Abel summable to fuzzy number

$$\mu(t) = \begin{cases} 1, & 0 \leq t \leq 2 \\ 0, & \text{otherwise,} \end{cases}$$

but is not Cesàro summable to any fuzzy number.

4 Comparison between Euler and Borel methods of summability of sequences of fuzzy numbers

Theorem 4.1. Let (u_n) be a sequence of fuzzy numbers. Then the sequence of q -th order Euler means of the p -th order Euler means of (u_n) is the sequence of $(p + q + pq)$ -th order Euler means of (u_n) .

Proof. Let (u_n) be a sequence of fuzzy numbers and (t_n^p) be the sequence of p -th order Euler means of (u_n) . Then the sequence of q -th order Euler means of (t_n^p) is

$$\begin{aligned}
 t_n^q(t_n^p) &= \frac{1}{(q+1)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} t_n^p \\
 &= \frac{1}{(q+1)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \frac{1}{(p+1)^k} \sum_{m=0}^k \binom{k}{m} p^{k-m} u_m \\
 &= \frac{1}{(q+1)^n} \sum_{m=0}^n u_m \sum_{k=m}^n \binom{n}{k} \binom{k}{m} q^{n-k} p^{k-m} \frac{1}{(p+1)^k} \\
 &= \frac{1}{(q+1)^n} \sum_{m=0}^n \binom{n}{m} \frac{1}{(p+1)^m} u_m \sum_{k=0}^{n-m} \binom{n-m}{k} q^{n-m-k} \left(\frac{p}{p+1}\right)^k \\
 &= \frac{1}{(q+1)^n} \sum_{m=0}^n \binom{n}{m} \frac{1}{(p+1)^m} \left(\frac{pq+p+q}{p+1}\right)^{n-m} u_m \\
 &= \frac{1}{(pq+p+q+1)^n} \sum_{m=0}^n \binom{n}{m} (pq+p+q)^{n-m} u_m = t_n^{pq+p+q},
 \end{aligned}$$

which completes the proof. \square

Theorem 4.2. If sequence (u_n) of fuzzy numbers is E_p summable to a fuzzy number μ , and $s > p > 0$, then it is E_s summable to μ .

Proof. Let $s > p > 0$ and let sequence (u_n) of fuzzy numbers be E_p summable to a fuzzy number μ . Then the sequence (t_n^p) of Euler means of (u_n) converges to μ . Besides it follows from Theorem 4.1 that $t_n^s = t_n^{\frac{s-p}{p+1}}(t_n^p)$. By regularity of Euler summability method we conclude that $(t_n^s) \rightarrow \mu$ and this completes the proof. \square

An E_s summable sequence is not necessarily E_p summable for $s > p > 0$, which can be seen by following example.

Example 4.3. Let (u_n) be a sequence of fuzzy number such that

$$u_n(t) = \begin{cases} 2 \sqrt[n]{t - (-p-s-1)^n}, & (-p-s-1)^n \leq t \leq (-p-s-1)^n + \frac{1}{2^n} \\ 1, & (-p-s-1)^n + \frac{1}{2^n} \leq t \leq (-p-s-1)^n + 1 \\ (-p-s-1)^n + 2 - t, & (-p-s-1)^n + 1 \leq t \leq (-p-s-1)^n + 2 \\ 0, & (\text{otherwise}) \end{cases}$$

Then α -level set of u_n is

$$[u_n]_\alpha = \left[(-p-s-1)^n + \left(\frac{\alpha}{2}\right)^n, (-p-s-1)^n + 2 - \alpha \right].$$

So α -level set of sequence t_n^s of s -th order Euler means is

$$[t_n^s]_\alpha = \left[\frac{1}{(s+1)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} \left\{ (-p-s-1)^k + \left(\frac{\alpha}{2}\right)^k \right\}, \frac{1}{(s+1)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} \{ (-p-s-1)^k + 2 - \alpha \} \right]$$

$$= \left[(-1)^n \frac{(p+1)^n}{(s+1)^n} + \left(\frac{2s+\alpha}{2s+2} \right)^n, (-1)^n \frac{(p+1)^n}{(s+1)^n} + 2 - \alpha \right].$$

Hence $D(t_n^s, \mu) = \frac{(p+1)^n}{(s+1)^n} + \left(\frac{2s+1}{2s+2} \right)^n \rightarrow 0$ where

$$\mu(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 2-t, & 1 \leq t \leq 2 \\ 0, & (\text{otherwise}), \end{cases}$$

from which we conclude that sequence (u_n) is E_s summable to fuzzy number μ . Now let investigate the E_p summability of (u_n) . α -level set of sequence t_n^p of p -th order Euler means is

$$\begin{aligned} [t_n^p]_\alpha &= \left[\frac{1}{(p+1)^n} \sum_{k=0}^n \binom{n}{k} p^{n-k} \left\{ (-p-s-1)^k + \left(\frac{\alpha}{2} \right)^k \right\}, \frac{1}{(p+1)^n} \sum_{k=0}^n \binom{n}{k} p^{n-k} \{ (-p-s-1)^k + 2 - \alpha \} \right] \\ &= \left[(-1)^n \frac{(s+1)^n}{(p+1)^n} + \left(\frac{2p+\alpha}{2p+2} \right)^n, (-1)^n \frac{(s+1)^n}{(p+1)^n} + 2 - \alpha \right], \end{aligned}$$

and then sequence (u_n) is not E_p summable to any number μ since the sequence $[t_n^p]_\alpha$ is not convergent.

Now we prove a lemma which is necessary to achieve the goal of this section.

Lemma 4.4. *Let $\sum_{n=0}^{\infty} u_n$ be a convergent series of fuzzy numbers. If $\sum_{n=0}^{\infty} x_n$ is a convergent series with non-negative real terms, then*

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n u_k \sum_{v=k}^n x_v = \sum_{k=0}^{\infty} u_k \sum_{v=k}^{\infty} x_v.$$

Proof. Let $\sum_{n=0}^{\infty} u_n$ be a convergent series of fuzzy numbers and $\sum_{n=0}^{\infty} x_n$ be a convergent series with non-negative real terms. Then we have

$$\begin{aligned} D \left(\sum_{k=0}^n u_k \sum_{v=k}^n x_v, \sum_{k=0}^{\infty} u_k \sum_{v=k}^{\infty} x_v \right) &= D \left(\sum_{k=0}^n u_k \sum_{v=k}^n x_v, \sum_{k=0}^n u_k \sum_{v=k}^n x_v + \sum_{k=0}^n u_k \sum_{v=n+1}^{\infty} x_v + \sum_{k=n+1}^{\infty} u_k \sum_{v=k}^{\infty} x_v \right) \\ &\leq D \left(\sum_{k=0}^n u_k \sum_{v=n+1}^{\infty} x_v, \bar{0} \right) + D \left(\sum_{k=n+1}^{\infty} u_k \sum_{v=k}^{\infty} x_v, \bar{0} \right) \\ &\leq \left\{ \sum_{v=n+1}^{\infty} x_v \right\} D \left(\sum_{k=0}^n u_k, \bar{0} \right) + \left\{ \sum_{v=0}^{\infty} x_v \right\} D \left(\sum_{k=n+1}^{\infty} u_k, \bar{0} \right). \end{aligned}$$

Since series $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} x_n$ are convergent, both of series are bounded and corresponding remainder terms converge to 0 as $n \rightarrow \infty$. So by limiting both sides of the expression above we get

$$\lim_{n \rightarrow \infty} D \left(\sum_{k=0}^n u_k \sum_{v=k}^n x_v, \sum_{k=0}^{\infty} u_k \sum_{v=k}^{\infty} x_v \right) = 0,$$

and the proof is completed. \square

Theorem 4.5. *If sequence (u_n) of fuzzy numbers is E_p summable to a fuzzy number μ , then it is Borel summable to μ .*

Proof. Let sequence (u_n) of fuzzy numbers be E_p summable to fuzzy number μ . Our aim is to show that $\sum_{n=0}^{\infty} \frac{x^n}{n!} u_n$ converges for all $x \in (0, \infty)$ and

$$\lim_{x \rightarrow \infty} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} u_n = \mu.$$

Since sequence (u_n) is E_p summable to μ , sequence a of Euler means converges to μ . Then we have $[t_n^p]_\alpha \rightarrow [\mu]_\alpha$ for $0 \leq \alpha \leq 1$, which, in special case, implies sequences $(u_n^-(0))$ and $(u_n^+(0))$ are E^p summable to $\mu_n^-(0)$ and $\mu_n^+(0)$, respectively. Then we have $u_n^-(0) = o((2p+1)^n)$ and $u_n^+(0) = o((2p+1)^n)$. So we get

$$D(u_n, \bar{0}) = \max\{|u_n^-(0)|, |u_n^+(0)|\} = o((2p+1)^n).$$

By using this fact, for all $x \in (0, \infty)$ we have

$$\sum_{n=0}^{\infty} D\left(\frac{x^n}{n!}u_n, \bar{0}\right) \leq \sum_{k=0}^{\infty} D(u_n, \bar{0}) \frac{x^n}{n!} \leq \sum_{n=0}^{\infty} \frac{((2p+1)x)^n}{n!} = e^{((2p+1)x)},$$

and by Theorem 2.6 the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}u_n$ converges for all $x \in (0, \infty)$. Besides we have

$$\begin{aligned} \sum_{n=0}^m (p+1)^n t_n^p \frac{x^n}{n!} &= \sum_{n=0}^m \frac{x^n}{n!} \sum_{k=0}^n \binom{n}{k} p^{n-k} u_k = \sum_{k=0}^m u_k \sum_{n=k}^m \binom{n}{k} \frac{x^n}{n!} p^{n-k} \\ &= \sum_{k=0}^m \frac{x^k}{k!} u_k \sum_{n=k}^m \frac{(px)^{n-k}}{(n-k)!} \end{aligned}$$

and by Lemma 4.4 we get

$$\sum_{n=0}^{\infty} t_n^p \frac{[(p+1)x]^n}{n!} = e^{px} \sum_{k=0}^{\infty} \frac{x^k}{k!} u_k.$$

Dividing both sides by $e^{(p+1)x}$ it follows that

$$\frac{1}{e^{(p+1)x}} \sum_{n=0}^{\infty} t_n^p \frac{[(p+1)x]^n}{n!} = e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} u_k.$$

Finally, since $(t_n^p) \rightarrow \mu$ and the Borel summability method is regular, by limiting both sides as $x \rightarrow \infty$ we conclude that

$$\lim_{x \rightarrow \infty} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} u_n = \mu.$$

□

On the other hand Borel summability of a sequence of fuzzy numbers may not imply E_p summability. This can be seen by the sequence (u_n) of fuzzy numbers defined by

$$u_n(t) = \begin{cases} t - (-1)^n n!, & (-1)^n n! \leq t \leq (-1)^n n! + 1 \\ \frac{(-1)^n n! + 2 - t}{t - (-1)^n n!}, & (-1)^n n! + 1 \leq t \leq (-1)^n n! + 2 \\ 0, & (\text{otherwise}). \end{cases}$$

Sequence (u_n) of fuzzy numbers is Borel summable to fuzzy number

$$\mu(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ \frac{2-t}{t}, & 1 \leq t \leq 2 \\ 0, & (\text{otherwise}), \end{cases}$$

but not E_p summable to any fuzzy number.

5 Conclusion

In this study we have proved comparison theorems for recently introduced summability methods of sequences of fuzzy numbers. Besides various results dealing with series of fuzzy numbers have been obtained. A comparison theorem, in general, provides us with the facility of extending the results of one method to another one directly without needing a separate proof. So it makes possible to utilize from the results in one method to achieve the goals related with the other method. In our case, in view of Theorem 3.3 and Theorem 4.5, we can extend the results for Abel summability method of sequences of fuzzy numbers [20] and Borel summability method of sequences of fuzzy numbers [21] to Cesàro and Euler summability methods, respectively. We mention some of these results concerning the convergence of summable sequences of fuzzy numbers below.

Corollary 5.1. *If sequence (u_n) of fuzzy numbers is Cesàro summable to fuzzy number μ and $nD(u_n, u_{n-1}) = o(1)$, then sequence (u_n) converges to μ .*

Corollary 5.2. *If series $\sum u_n$ of fuzzy numbers is Cesàro summable to fuzzy number ν and $nD(u_n, \bar{0}) = o(1)$, then $\sum u_n = \nu$.*

Corollary 5.3. [22] *If sequence (u_n) of fuzzy numbers is E_p summable to fuzzy number μ and $\sqrt{n}D(u_{n-1}, u_n) = o(1)$, then (u_n) converges to μ .*

Corollary 5.4. [22] *If series $\sum u_n$ of fuzzy numbers is E_p summable to fuzzy number ν and $\sqrt{n}D(u_n, \bar{0}) = o(1)$, then $\sum u_n = \nu$.*

References

- [1] H. Altınok, R. Çolak, Y. Altın, On the class of λ -statistically convergent difference sequences of fuzzy numbers, *Soft Computing* 16(6) (2012), 1029–1034.
- [2] Y. Altın, M. Mursaleen, H. Altınok, Statistical summability $(C, 1)$ for sequences of fuzzy real numbers and a Tauberian theorem, *Journal of Intelligent and Fuzzy Systems* 21(6) (2010), 379–384.
- [3] B. Bede, S. G. Gal, Almost periodic fuzzy-number-valued functions, *Fuzzy Sets and Systems* 147 (2004), 385–403.
- [4] İ. Çanak, On the Riesz mean of sequences of fuzzy real numbers, *Journal of Intelligent and Fuzzy Systems* 26(6) (2014), 2685–2688.
- [5] İ. Çanak, Tauberian theorems for Cesàro summability of sequences of fuzzy numbers, *Journal of Intelligent and Fuzzy Systems*, 27(2) (2014), 937–942.
- [6] İ. Çanak, On Tauberian theorems for Cesàro summability of sequences of fuzzy numbers, *Journal of Intelligent and Fuzzy Systems* 30(5) (2016), 2657–2662.
- [7] Y. K. Kim, B. M. Ghil, Integrals of fuzzy-number-valued functions, *Fuzzy Sets and Systems* 86 (1997), 213–222.
- [8] M. Matloka, Sequences of fuzzy numbers, *Busefal* 28 (1986), 28–37.
- [9] M. Mursaleen, H. M. Srivastava, S. K. Sharma, Generalized statistically convergent sequences of fuzzy numbers, *Journal of Intelligent and Fuzzy Systems* 30(3) (2016), 1511–1518.
- [10] S. Nanda, On sequence of fuzzy numbers, *Fuzzy Sets and Systems* 33 (1989), 123–126.
- [11] S. A. Sezer, İ. Çanak, Power series methods of summability for series of fuzzy numbers and related Tauberian Theorems, *Soft Computing* (2015). doi: 10.1007/s00500-015-1840-0

- [12] M. Stojaković, Z. Stojaković, Addition and series of fuzzy sets, *Fuzzy Sets and Systems* 83 (1996), 341–346.
- [13] M. Stojaković, Z. Stojaković, Series of fuzzy sets, *Fuzzy Sets and Systems* 160 (2009), 3115–3127.
- [14] P. V. Subrahmanyam, Cesàro summability of fuzzy real numbers, *Journal of Analysis* 7 (1999), 159–168.
- [15] Ö. Talo, F. Başar, Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations, *Computers and Mathematics with Applications* 58(4) (2009), 717–733.
- [16] Ö. Talo, C. Çakan, On the Cesàro convergence of sequences of fuzzy numbers, *Applied Mathematics Letters* 25 (2012), 676–681.
- [17] Ö. Talo, F. Başar, On the Slowly Decreasing Sequences of Fuzzy Numbers, *Abstract and Applied Analysis* 2013 (2013), 1–7.
- [18] B. C. Tripathy, A. Baruah, Nörlund and Riesz mean of sequences of fuzzy real numbers, *Applied Mathematics Letters* 23 (2010), 651–655.
- [19] B. C. Tripathy, M. Sen, On fuzzy I-convergent difference sequence space, *Journal of Intelligent and Fuzzy Systems* 25(3) (2013), 643–647.
- [20] E. Yavuz, Ö. Talo, Abel summability of sequences of fuzzy numbers, *Soft Computing* 20(3) (2016), 1041–1046.
- [21] E. Yavuz, H. Çoşkun, On the Borel summability method of sequences of fuzzy numbers, *Journal of Intelligent and Fuzzy Systems* 30(4) (2016), 2111–2117.
- [22] E. Yavuz, Euler summability method of sequences of fuzzy numbers and a Tauberian theorem, *Journal of Intelligent and Fuzzy Systems* (2016). doi:10.3233/JIFS-161429
- [23] L. A. Zadeh, Fuzzy sets, *Information and Control* 8 (1965), 29–44.